Lecture 5. Recall DoC of $\sum_\alpha a_\alpha z^\alpha$ is
$D = \{ z_0 : \sum a_\alpha z^\alpha$ conv. abs. for all z in open nbhd of $z_0 \}$.

Thm1 Assume $D \neq \emptyset$. The following hold.

(i) D = interior of $\{ z_0 : |a_\alpha z^\alpha| \leq C$ for all α and some $C \}$ and the convergence in D is normal ($\Rightarrow$ P.S. conv. to a holom. fcn in D).

(ii) D is Reinhardt, i.e. invariant under $t = (t_1, \dots, t_n) \to (e^{it_1} z_1, \dots, e^{it_n} z_n)$.

(iii) If $D^* = \{ \xi \in \mathbb{R}^n : (e^{\xi_1}, \dots, e^{\xi_n}) \in D \}$ open
(essentially $z \in D \Leftrightarrow \log|z| \in D^*$), then $D^* \subseteq \mathbb{R}^n$ is convex, $\xi \in D^* \Rightarrow \eta \in D^*$ provided that $\eta_j \leq \xi_j$, $j = 1, \dots, n$, and $z \in D \Leftrightarrow |z_j| \leq e^{\xi_j}$, $j = 1, \dots, n$, for some $\xi \in D^*$.

Def Reinhardt domains Ω satisfying prop's in 1iii) are called logarithmically convex.

Thm 2. Let Ω be a Reinhardt domain s.t. $0 \in \Omega$.

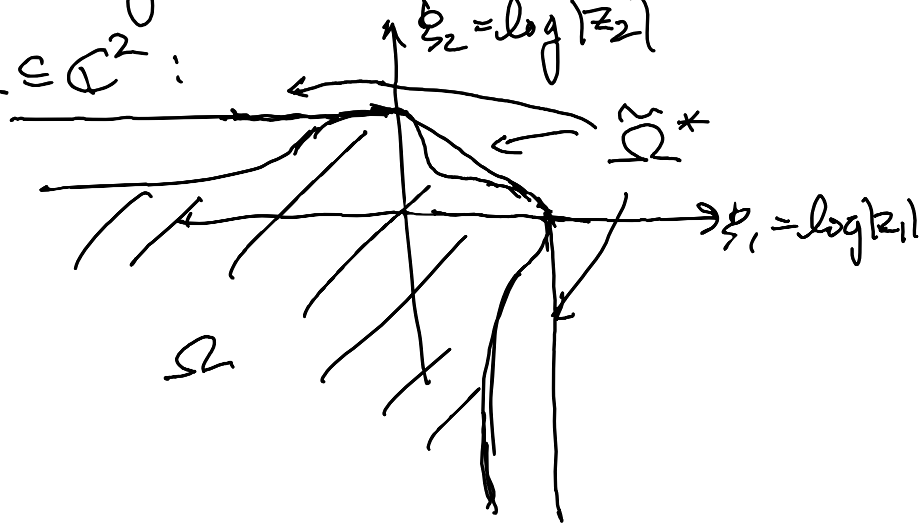
(i) If $f \in \mathcal{O}(\Omega)$ then f has a unique P.S.

space of holom. fcns

that converges normally to f in Ω.

(ii) The P.S. of f converges normally in $\tilde{\Omega}$, the smallest log-convex Reinhardt domain containing Ω. $(\Rightarrow \exists\, \tilde{f} \in \mathcal{O}(\tilde{\Omega})$ s.t. $\tilde{f}|_{\Omega} = f)$.

"Ex." $\Omega \subseteq \mathbb{C}^2$:



For pfs, see Hörmander Sec 2.4.

Domains of Holomorphy.

Motivated by Hartogs and the log convex nature of domains of convergence, we make the:

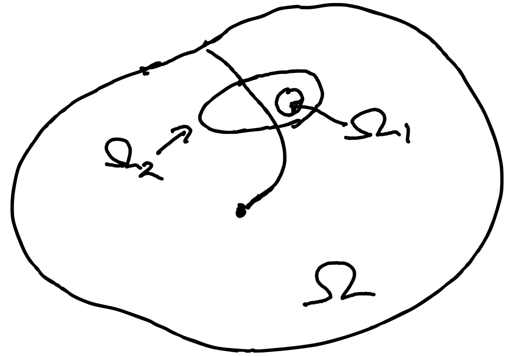
Def. A domain $\Omega \subseteq \mathbb{C}^n$ is a Domain of Holomorphy (DoH) if $\nexists$ nonempty open sets Ω_1, Ω_2 s.t.

(a) $\Omega_1 \subseteq \Omega \cap \Omega_2$

(b) Ω_2 connected and $\Omega \not\subseteq \Omega_2$.

(c) $\forall u \in \mathcal{O}(\Omega)\ \exists u_2 \in \mathcal{O}(\Omega_2)$ s.t. $u = u_2$ on Ω_1.

While we would like to say Ω is a DoH if there is no larger Ω' s.t. all holom. fcns extend to Ω' (as for Reinhardt domains), this does not suffice. Consider a situation like:



To achieve a characterization of DoH's, we introduce

<u>Def.</u> Let $K \subset\subset \Omega$. The <u>$\mathcal{O}(\Omega)$-hull</u> $\hat{K}_\Omega$ of K is given by

$$\hat{K}_\Omega = \{ z \in \Omega : |u(z)| \leq \sup_K |u| \}$$

<u>Thm 3</u> Let $\Omega \subseteq \mathbb{C}^n$ domain. TFAE:

(i) Ω is a DoH.

(ii) $K \subset\subset \Omega \Rightarrow \hat{K}_\Omega \subset\subset \Omega$ and

$$d(K, \partial\Omega) = d(\hat{K}_\Omega, \partial\Omega)$$

distance to $\partial\Omega$: $\inf_{z \in K, w \in \partial\Omega} |z-w|$ (or any other distance)

(iii) $\exists\, u \in \mathcal{O}(\Omega)$ that cannot be extended past Ω, i.e. $\nexists\, \Omega_1, \Omega_2$ as in DoH definition for u.